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1997 J. Phys. A: Math. Gen. 30 2281

(http://iopscience.iop.org/0305-4470/30/7/010)

J. Phys. A: Math. Gen. **30** (1997) 2281–2285. Printed in the UK PII: S0305-4470(97)77878-8

Asymptotic results for the number of multidimensional partitions of an integer and directed compact lattice animals

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Received 16 September 1996, in final form 31 December 1996

Abstract. There is an exact one-to-one correspondence between the number of $(d-1)$-dimensional partitions of an integer and the number of directed compact lattice animals in d dimensions. Using enumeration techniques, we obtain upper and lower bounds for the number of multidimensional partitions (both restricted and unrestricted). We show that asymptotically the number of unrestricted $(d-1)$-dimensional partitions of an integer n goes as $\exp(Cn^{(d-1)/d})$. We also show that for restricted partitions in $(d-1)$ dimensions (with j dimensions finite, $0 < j < d-1$), this number goes as $\exp(Cn^{(d-j-1)/(d-j)}(\prod_{k=1}^{j} L_k)^{1/(d-j)})$, where L_k is the extent of the lattice along the kth axis.

Partitioning of integers is a problem which has been extensively studied in number theory [1–4]. A linear or one-dimensional partition† of a positive integer n is given by

$$n = n_1 + n_2 + \cdots + n_k \tag{1}$$

where all n_i's are positive integers and $n_i \geqslant n_{i+1}$. The partition is called an unrestricted partition if there are no other restrictions on the value of n_i and of k. The number of such unrestricted partitions $p(n)$ is given by the Hardy–Ramanujam [5] result

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right). \tag{2}$$

In a similar manner, one can write a planar or two-dimensional partition as

$$n = \sum_{i=1}^{k_1} \sum_{j=1}^{k_2} n_{ij}. \tag{3}$$

Again the n_{ij}'s are positive integers and we have the restriction $n_{i-1,j} \geqslant n_{ij}$ and $n_{i,j-1} \geqslant n_{ij}$. One may, in a straightforward manner, extend these definitions for partitions into dimensions greater than two. Moment generating functions as well as the total number of restricted as well as unrestricted partitions are known for dimensions 1 and 2 [5, 6]. However, the problem of the number of partitions in dimensions greater than two is an unsolved problem in number theory [4–8].

† Some authors (see for example [4]) call this a two-dimensional partition.

0305-4470/97/072281+05$19.50

In a recent paper, Wu *et al* [5] proposed a new kind of lattice animal called the directed compact lattice animal (DCLA) which is constructed as follows. Consider a lattice point $\bar{k} = (k_1, k_2, \ldots, k_d)$ in d dimensions. This point can only be occupied if each of the lattice points $\bar{k} - \bar{e}_i, i = 1, 2, \ldots, d$, is occupied where $\bar{e}_i$ are unit vectors along the d axes. The construction of the DCLA begun by assuming that all the points (in the positive quadrant) along the d hyperplanes of $(d-1)$ dimensions perpendicular to the axes are occupied. It is easy to see that there is an exact one-to-one correspondence between the number of DCLAs of n points in d dimensions and the number of $(d-1)$-dimensional partitions of n. Based on the results for one- and two-dimensional partitions, Wu *et al* [5] conjectured that the number $A_d(n; L_1, L_2, \ldots, L_d)$ of d-dimensional DCLAs with $(d \geqslant 2)$ of n points goes asymptotically as

$$A_d(n; L_1, L_2, \ldots, L_d) \sim \exp(cn^{(d-1)/d}) \tag{4}$$

where L_i is the extent of the lattice along the ith axis, provided

$$\prod_{i=1}^{d} L_i \sim n/2. \tag{5}$$

In this paper, we establish upper and lower bounds on A_d to prove the following two results.

(i) For infinite lattices and n sufficiently large

$$C_1 \leqslant \frac{\log A_d(n)}{n^{(d-1)/d}} \leqslant C_2 \tag{6}$$

where C_1 and C_2 (throughout this paper) are general symbols for positive finite constants independent of n but dependent on d.

(ii) For lattices which are infinite in extent in at least two dimensions and finite in extent in j dimensions

$$C_1 \leqslant \frac{\log A_d(n; L_1, L_2, \ldots, L_j)}{n^{(d-j-1)/(d-j)}(\prod_{j'=1}^{j} L_{j'})^{1/(d-j)}} \leqslant C_2 \tag{7}$$

for n sufficiently large and $L_{j'} \leqslant n^{1/d}$, $j' = 1, 2, \ldots, j$, and $0 < j < d-1$.

The first result is similar to the conjecture of Wu *et al* [5]. However, the condition given by equation (5) is not sufficient. We need to impose a stronger condition namely all the L_j's are greater than equal to $n^{1/d}$ (as shown by the second result).

We now prove the first result by obtaining bounds for infinite lattices.

1. Lower bound

Consider all DCLAs formed as follows. Let $\bar{k} = (k_1, k_2, \ldots, k_d)$ be a lattice point in d dimensions. Let all the lattice points $\bar{k}$ such that

$$\begin{aligned} &0 < k_i \leqslant m \qquad \text{for } i = 1, 2, \ldots, d-1 \\ &0 < k_d \leqslant m_{\bar{k}} \end{aligned} \tag{8}$$

be occupied.

Here $m_k = \alpha - \sum_{l=1}^{d-1} k_l + \Psi_{\bar{k}}$. Here, $\alpha = (d-1)m$ and $\Psi_{\bar{k}}$'s take on values 0 or 1 and m is given by the equation

$$m^{d-1} + (d-1)\frac{m^{d-1}(m+1)}{2} = n. \tag{9}$$

All other lattice points are unoccupied. Clearly all such constructions are DCLAs. Furthermore, the value of m has been so chosen that if all $\Psi_{\bar{k}}$'s are equal to 1, the number of lattice points occupied is equal to n. This equation, in general, does not have a solution in integers, but the truncation involved in taking m as an integer is of no significance for the present study. In fact, it suffices for our purpose to consider only the leading term on the left-hand side of equation (9) and take m as $(2n/(d-1))^{1/d}$. However, the number of points occupied for different combinations of $\Psi_{\bar{k}}$'s is, in general, different. This number n' satisfies the inequality $n_1 \leqslant n' \leqslant n$, with $n_1 = n - m^{(d-1)}$.

The total number of DCLAs formed by this procedure is clearly $2^{m^{(d-1)}}$. Since this procedure does not count all possible DCLAs, we have the inequality

$$\sum_{n'=n_1}^{n} A_d(n') \geqslant 2^{m^{d-1}}. \tag{10}$$

But $A_d(n)$ is a monotonically increasing function of n. We, therefore, have the inequality

$$(n - n_1)A_d(n) \geqslant 2^{m^{d-1}}. \tag{11}$$

Using the fact that $m = \mathrm{O}(n^{1/d})$, we obtain, for large enough n the asymptotic inequality

$$\log[A_d(n)]/n^{(d-1)/d} \geqslant C_1. \tag{12}$$

2. Upper bound

We get the upper bound by induction on d. The result is true for $d = 3$ [5]. To use the method of induction, we try to express the d-dimensional DCLA of n points as a combination of $(d-1)$-dimensional DCLAs whose total number is $\mathrm{O}(n^{1/d})$. It is not possible to do this in all cases using only parallel hyperplanes, since a d-dimensional DCLA of n points can occupy points greater than $n^{1/d}$ in every one of the d dimensions. We, therefore, use a more elaborate procedure. From the d-dimensional DCLA, we first delete the $(d-1)$-dimensional hyperplane (perpendicular to any one of the axes) containing the largest number of points. Clearly, this will be one of the outermost hyperplanes, that is, one of the hyperplanes $x_1 = 1$ or $x_2 = 1$ or $\ldots$ $x_d = 1$. The configuration left out after this deletion is still a DCLA and we repeat the procedure until all points of the original DCLA are deleted. Clearly, the number of steps for the whole process is less than equal to $dn^{1/d} (:= m_2)$. This is because, if any point is left out after m_2 steps the coordinates of the point must satisfy the inequalities $x_i > m_1$ for each of $i = 1, 2, \ldots, d$, where $m_1 = n^{1/d}$. By the definition of a DCLA, this implies that all the points inside the hypercube $0 \leqslant x_i \leqslant m_1$, $i = 1, 2, \ldots, d$, are occupied which, in turn implies that the DCLA contains more than n points.

Now, we construct a set S of lattice animals (not all of them DCLAs) using the following procedure. First, we write down all possible one-dimensional partitions of the number n into k partitions with $k \leqslant m_2$, where $m_2 = d.n^{1/d}$. Let one such partition be

$$n = n_1 + n_2 + \cdots + n_k \tag{13}$$

with $n_i \geqslant n_{i+1}$ for $i = 1, 2, \ldots, k-1$.

We set all the numbers $n_{k'} = 0$ for $k' = k+1, k+2, \ldots, m_2$. Now, we construct DCLAs of $n_1, n_2, \ldots, n_{m_2}$ points respectively along the m_2 hyperplanes ($(d-1)$-dimensional) $x_1 = 1$, $x_1 = 2, \ldots, x_1 = m_1, x_2 = 1, x_2 = 2, \ldots, x_2 = m_1, \ldots, x_d = 1, x_d = 2, \ldots, x_d = m_1$ such that the number of points on the DCLA in $x_j = l_1$ is greater than or equal to that in the hyperplane $x_j = l_2$ if $l_1 < l_2$.

Clearly, this set S contains all the d-dimensional DCLAs of n points (and many other configurations which are not DCLAs). Therefore, $A_d(n)$, the number of d-dimensional DCLAs of n points is less than the cardinality $|S|$ of S:

$$A_d(n) \leqslant |S|. \tag{14}$$

Now, we can obtain a bound for $|S|$, as a product of three factors.

(i) The number of one-dimensional partitions of n with the number of partitions not exceeding m_2. This is less than the total number of unrestricted partitions of n.

(ii) $\prod_{i=1}^{m_2} A_{d-1}(n_i)$ which is the product of the number of DCLAs in $(d-1)$ dimensions for each of the numbers n_i.

(iii) The number of ways the m_2 hyperplanes are chosen. This number is less than d^{m_2}, since at each step there are at most d choices of choosing a hyperplane. Therefore,

$$A_d(n) \leqslant \sum \prod_{i=1}^{m_2} A_{d-1}(n_i) d^{m_2}. \tag{15}$$

The sum extends over all positive integral values of n_i such that $n_i \geqslant n_{i+1}$ and $\sum_{i=1}^{m_2} n_i = n$. The largest value of the product in (15) occurs when all the n_i's are equal to $n/m_2 = n^{(d-1)/d}/d$. We note that the number of terms in the sum is less than or equal to $\exp(c\sqrt{(n)})$ and also that d^{m_2} is of a lower order. Using the induction hypothesis, that is $\log(A_{d-1}(n))/n^{(d-2)/(d-1)} \leqslant C_2$, we have the following asymptotic inequality for large enough n:

$$\frac{\log(A_d(n))}{n^{(d-1)/d}} \leqslant C_2. \tag{16}$$

3. Finite lattices

The proof for equation (7) for finite lattices is very similar to that for infinite lattices. We consider a lattice which has j sides of lengths $L_i \leqslant n^{1/d}, i = 1, 2, \ldots, j$, and the others are infinite in extent. j, here, must be less than $(d-1)$. To get a lower bound, as before, we construct DCLAs satisfying the conditions given in equations (8–10) except that now

$$k_i \leqslant L_i \qquad i = 1, 2, \ldots, j \tag{17}$$

$$k_i \leqslant \mu \qquad \text{for } i = j+1, \ldots, d-1 \tag{18}$$

$$k_d \leqslant \mu_{k'} \tag{19}$$

and μ is given by

$$L\mu^{d-j-1} + \mu^{d-j-1} L \sum_1^j \frac{(L_i + 1)}{2} + \frac{L(d-j-1)\mu^{d-j-1}(\mu+1)}{2} = n. \tag{20}$$

Here, $L = \prod_1^j L_i$ and $\mu_{\bar{k}} = \alpha' - \sum_1^{d-1} k_l$ with $\sum_1^j L_i + (d-j-1)'m$. From equation (20), we see that $\mu = \mathrm{O}((n/L)^{1/(d-j)})$.

Following a procedure similar to that used for the derivation of equation (12), we get, for n large enough, the inequality

$$\frac{\log[A_d(n, L_1, \ldots, L_j)]}{(n^{(d-j-1)/(d-j)} L^{1/(d-j)})} \geqslant C_1. \tag{21}$$

To obtain an upper bound for $d - 2 \geqslant j \geqslant 1$ we note that we have, for $j = 0, d \geqslant 2$, the inequality (equation (16))

$$\frac{\log(A_d(n))}{n^{(d-1)/d}} \leqslant C_2. \tag{22}$$

Now, for $j = 1$ the lattice is infinite in all directions except one in which it is $L_1(\leqslant n^{1/d})$. We can, therefore, express the d-dimensional DCLA as a combination of L_1 unrestricted DCLAs each of $(d-1)$ dimensions. We can then write an equation similar to (15):

$$A_d(n, L_1) \leqslant \sum \prod_{i=1}^{L_1} A_{d-1}(n_i). \tag{23}$$

Using equation (23) and noting that the maximum occurs when each of the L_1 DCLAs contains $n_1(= n/L_1)$ points, we get the inequality for n large enough and $j = 1$:

$$\frac{\log\,(A_d(n; L_1))}{n^{(d-2)/(d-1)} L_1^{1/(d-1)}} \leqslant C_2. \tag{24}$$

The procedure can be extended in a straightforward manner to higher values of j to obtain

$$\frac{\log\,[A_d(n, L_1, \ldots, L_j)]}{n^{(d-j-1)/(d-j)} L^{1/(d-j)}} \leqslant C_2. \tag{25}$$

Combining equations (21) and (25) we obtain equation (7).

Because the starting value is $d = 2$, $j = 0$, the inequality is valid only for values of $j \leqslant d-2$. From equations (9) and (10) of [5], we see that for the case $d = 2$, $j = 1$, if L_1 is less than $n^{1/d}$, (25) is not directly applicable; the denominator has to be multiplied by an additional log factor.

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